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ASYMPTOTIC ANALYSIS OF THE STRUCTURE OF LONG-WAVE GÖRTLER VORTICES IN A HYPERSONIC BOUNDARY LAYER

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An asymptotic (at high Reynolds and Görtler numbers) model of nonlinear long-wave Görtler vortices localized inside the boundary layer near a concave surface located in a hypersonic viscous gas flow in the regime of weak viscid–inviscid interaction is constructed. The maximum wavelength is evaluated. Numerical solutions are obtained for an inviscid local limit in the linear approximation. It is shown that an increase in the free-stream Mach number exerts a stabilizing effect on the vortices, and a change in the Prandtl number has no significant effect on them. For the case where the vortices form a three-layered disturbed flow structure, it is shown analytically for the first time that surface heating exerts a stabilizing action on the vortices.

An analysis of the Navier–Stokes equations at high Reynolds and Görtler numbers allowed construction of an asymptotic theory of Görtler vortices [1] in the boundary layer of a fluid near a concave surface [2–8]. In studying the Görtler vortices in a compressible boundary layer, one should take into account the effect of various factors, such as the free-stream Mach number, the surface temperature, the Prandtl number, and the physicochemical processes in the gas. Of special interest is the study of the influence of surface cooling on vortex dynamics [9–12], since the neglect of this parameter does not allow one to ensure the necessary strength of constructions of flying vehicles at hypersonic velocities. El-Hady and Verma [10] noted a weak stabilizing effect of surface cooling on vortices, whereas the authors of [9, 11, 12] believed that surface cooling has a destabilizing influence on vortices. The destabilizing action of surface cooling was supported by studying the dynamics of short-wave vortices [13], and an explanation of this phenomenon was proposed. The mechanism of the stabilizing effect of increasing Mach number on the vortices was described in [14].

The dynamics of long-wave Görtler vortices near a concave surface exposed to a hypersonic gas flow is studied in the present paper in the linear approximation in the regime of weak viscid–inviscid interaction at high Reynolds and Görtler numbers.

1. Let a concave surface be exposed to a uniform viscous gas flow. It is assumed that its dimensionless curvature is small: $k = L/R \ll 1$ (R is the radius of surface curvature and L is the distance from the leading edge of the surface to the point of incipience of vortices), the free-stream Mach numbers is $\mathrm{M}_\infty \gg 1$, and the Reynolds number is high ($\mathrm{Re}_\infty = \rho_\infty u_\infty L/\mu_\infty \gg 1$), but the laminar–turbulent transition has not yet occurred. Here ρ_∞, u_∞, and μ_∞ are the free-stream density, velocity, and viscosity of the gas, respectively. Hereinafter, all linear dimensions are normalized to L, the pressure p and enthalpy h to $\rho_\infty u_\infty^2$ and u_∞^2, respectively, and the remaining parameters to their free-stream values.

It is also assumed that pressure perturbations due to the displacing action of the boundary layer Δp_δ and the surface curvature Δp_k are small as compared to the free-stream gas pressure:

$$\Delta p_\delta \sim \delta/\mathrm{M}_\infty \ll 1/\mathrm{M}_\infty^2, \qquad \Delta p_k \sim k/\mathrm{M}_\infty \ll 1/\mathrm{M}_\infty^2$$

(δ is the boundary-layer thickness). In the boundary layer with characteristic dimensions $\Delta x \sim 1$ and $\Delta y \sim \delta$ (the x axis is directed along the surface and the y axis is normal to it), the stream functions obey the estimates for the regime of weak hypersonic viscid–inviscid interaction [15]

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$$u \sim h \sim 1, \qquad v \sim \delta, \qquad p \sim \rho \sim 1/\mathrm{M}_\infty^2, \qquad \mu \sim \mathrm{M}_\infty^2, \qquad \delta \sim \mathrm{M}_\infty^2/\mathrm{Re}_\infty^{1/2}, \tag{1.1}$$

where u and v are the velocity components along the x and y axes, respectively. In obtaining these estimates, we used the linear dependence of viscosity on enthalpy

$$\mu = A\mathrm{M}_\infty^2 h \tag{1.2}$$

and the equation of state of a perfect gas

$$\gamma p = (\gamma - 1)\rho h, \tag{1.3}$$

where A is a constant and γ is the ratio of specific heats.

We introduce a vertical coordinate of the Lees–Dorodnitsyn type

$$n(x, y) = \frac{1}{\sqrt{2x}} \int\limits_0^y \rho\, dy. \tag{1.4}$$

Using estimates (1.1) and relations (1.2) and (1.3), we write the known self-similar boundary-layer equations

$$u'' + \varphi u' = 0, \qquad \varphi = \int\limits_0^n u\, dn, \qquad \frac{h''}{\mathrm{Pr}} + \varphi h' = -u'^2,$$
$$u(0) = \varphi(0) = 0, \quad h(0) = h_w \quad \text{or} \quad h'(0) = 0, \quad u(\infty) = 1, \quad h(\infty) = 1/((\gamma - 1)\mathrm{M}_\infty^2), \tag{1.5}$$

where $(\,\cdot\,)' = d/dn$, the subscript w refers to quantities on the concave surface, and the Prandtl number Pr is assumed to be constant. In solving the boundary-value problem (1.5), self-similar profiles of the streamwise component of velocity $u_0(n)$ and enthalpy $h_0(n)$ in the undisturbed boundary layer in the cross section $x_0 \sim 1$ were obtained.

2. It is known that a two-dimensional laminar boundary layer on a concave surface may lose stability as the critical Görtler number $\mathrm{G}_\infty = 2(\mathrm{Re}_\infty^{1/2}/\mathrm{M}_\infty^2)(L/R)$ is exceeded [1]. Then, streamwise stationary Görtler vortices appear inside the boundary layer, and a two-dimensional flow becomes three-dimensional. Below we study the development of such vortices with a wavelength greater than the boundary-layer thickness at high values of the Görtler number $\mathrm{G}_\infty \sim æ/\delta \gg 1$ ($k = æK$, $K \sim 1$, and $\delta \ll æ \ll 1$), where the vortices are definitely absent.

We consider a disturbed vortex region of the flow with characteristic transverse dimensions $\Delta y \sim \delta$ and $\Delta z \gg \delta$ (the z axis is perpendicular to the xy plane) at a finite distance $x_0 \sim 1$ from the leading edge of the concave surface. It is assumed that the disturbed region occupies the entire boundary layer, the vortices are localized inside this region, and the stream functions [see Eq. (1.1)] have the following orders:

$$u \sim h \sim 1, \qquad p \sim \rho \sim 1/\mathrm{M}_\infty^2, \qquad \mu \sim \mathrm{M}_\infty^2. \tag{2.1}$$

In constructing multilayered flow patterns, this region being the main part of the boundary layer is usually indicated in the literature by 2 [16], the weakly disturbed external region of a uniform incoming flow is indicated by 1, and the internal near-wall part of the boundary layer is indicated by 3.

Let vortex formation cause nonlinear disturbances in the boundary layer (for example, $\Delta u \sim u \sim 1$). In this case, in the field of centrifugal forces, there appears an additional pressure perturbation Δp, which induces the velocity component w in the direction of the z axis:

$$\Delta p \sim k\rho u^2 \Delta y \sim æ\delta/\mathrm{M}_\infty^2, \qquad w \sim (\Delta p/\rho)^{1/2} \sim (æ\delta)^{1/2}. \tag{2.2}$$

Since the disturbed vortex region is essentially three-dimensional, the continuity equation yields estimates for the characteristic transverse size of this region Δz and the vertical component of velocity v

$$\Delta z \sim w\Delta x/u \sim (æ\delta)^{1/2}\Delta x, \qquad v \sim u\Delta y/\Delta x \sim \delta/\Delta x, \tag{2.3}$$

where $(\delta/æ)^{1/2} \ll \Delta x \leqslant 1$ is the characteristic longitudinal scale of the disturbed vortex region. For $\Delta x \sim 1$, Eq. (2.3) yields an estimate for the maximum characteristic transverse size of the disturbed vortex region (or the maximum wavelength of the Görtler vortices in the gas):

$$\Delta z_* \sim (æ\delta)^{1/2}.$$

A comparison of the orders of magnitude of convective and dissipative terms of the Navier–Stokes equations [$\rho u u_x \sim (\mu u_y)_y/\mathrm{Re}_\infty$] shows that viscosity in the disturbed vortex region should be taken into account only for $\Delta x \sim 1$; for $\Delta x \ll 1$, viscous effects are insignificant.

In considering three-dimensional disturbed regions of the boundary layer with characteristic transverse dimensions $\Delta z \gg \delta$, it is necessary to take into account their possible interaction with weakly disturbed external regions of the uniform incoming flow, where $\rho \sim u \sim 1$ (see, e.g., [17]). For the interaction of such regions, it is necessary that the order of magnitude of the vertical component of velocity v remain constant [16]. In this case, the external region is region 1 with characteristic dimensions $(\delta/æ)^{1/2} \ll \Delta x \leqslant 1$ and $\Delta y \sim \Delta z \sim (æ\delta)^{1/2}\Delta x \gg \delta$. The estimates show that a pressure perturbation $\Delta p \sim \rho uv\Delta y/\Delta x \sim æ^{1/2}\delta^{3/2}/\Delta x$ is induced in this region; for $\Delta x \gg (\delta/æ)^{1/2}$, this perturbation is smaller in order of magnitude than that induced near the external boundary of the disturbed vortex region 2 [see (2.2)]. This means that disturbances induced in the disturbed vortex region 2 decay in the external region 1, and there is no reverse effect of disturbances from region 1 on region 2.

Estimates (2.1)–(2.3) allow introduction of new variables $x = x_0 + \Delta x x_2$, $y = \delta y_2$, and $z = (æ\delta)^{1/2}\Delta x z_2$ and asymptotic expansions of the stream functions for the disturbed spatial vortex region 2:

$$u = u_2 + \ldots, \qquad v = (\delta/\Delta x)v_2 + \ldots, \qquad w = (æ\delta)^{1/2}w_2 + \ldots,$$
$$p = \frac{1}{\gamma \mathrm{M}_\infty^2} + \ldots + \frac{æ\delta}{\mathrm{M}_\infty^2}\,p_2 + \ldots, \qquad \rho = \frac{\rho_2}{\mathrm{M}_\infty^2} + \ldots, \qquad h = h_2 + \ldots, \qquad \mu = \mathrm{M}_\infty^2\mu_2 + \ldots . \tag{2.4}$$

Hereinafter, insignificant terms in the expansion for p are omitted, which does not affect the transverse component of velocity w.

Substituting expansions (2.4) into the Navier–Stokes equations and into Eqs. (1.2) and (1.3) and performing the limiting transition

$$\mathrm{M}_\infty \to \infty, \qquad \delta \to 0, \qquad æ \to 0, \qquad \mathrm{M}_\infty\delta \to 0, \qquad \mathrm{M}_\infty æ \to 0, \qquad \delta \ll æ \ll 1, \tag{2.5}$$

we find that the flow in region 2 in the first approximation is described by the system

$$\frac{\partial(\rho_2 u_2)}{\partial x_2} + \frac{\partial(\rho_2 v_2)}{\partial y_2} + \frac{\partial(\rho_2 w_2)}{\partial z_2} = 0,$$
$$\rho_2\Big(u_2\,\frac{\partial u_2}{\partial x_2} + v_2\,\frac{\partial u_2}{\partial y_2} + w_2\,\frac{\partial u_2}{\partial z_2}\Big) = \Delta x\,\frac{\partial}{\partial y_2}\Big(\mu_2\,\frac{\partial u_2}{\partial y_2}\Big),$$
$$\frac{\delta}{æ}\,\rho_2\Big(u_2\,\frac{\partial v_2}{\partial x_2} + v_2\,\frac{\partial v_2}{\partial y_2} + w_2\,\frac{\partial v_2}{\partial z_2}\Big) + \Delta x^2\Big(K\rho_2 u_2^2 + \frac{\partial p_2}{\partial y_2}\Big) = 0, \tag{2.6}$$
$$\rho_2\Big(u_2\,\frac{\partial w_2}{\partial x_2} + v_2\,\frac{\partial w_2}{\partial y_2} + w_2\,\frac{\partial w_2}{\partial z_2}\Big) + \frac{\partial p_2}{\partial z_2} = \Delta x\,\frac{\partial}{\partial y_2}\Big(\mu_2\frac{\partial w_2}{\partial y_2}\Big),$$
$$\rho_2\Big(u_2\,\frac{\partial h_2}{\partial x_2} + v_2\,\frac{\partial h_2}{\partial y_2} + w_2\,\frac{\partial h_2}{\partial z_2}\Big) = \Delta x\Big[\frac{1}{\mathrm{Pr}}\,\frac{\partial}{\partial y_2}\Big(\mu_2\,\frac{\partial h_2}{\partial y_2}\Big) + \mu_2\Big(\frac{\partial u_2}{\partial y_2}\Big)^2\Big],$$
$$(\gamma - 1)\rho_2 h_2 = 1, \qquad \mu_2 = Ah_2.$$

For $\Delta x \sim 1$, when the characteristic dimensions of the boundary layer and disturbed vortex region 2 have the same order of magnitude and viscous effects should be taken into account, usual no-slip and adhesion conditions are fulfilled on the concave surface:

$$u_2 = v_2 = w_2 = 0, \qquad h_2 = h_{2w} \quad \text{or} \quad \frac{\partial h_2}{\partial y_2} = 0 \qquad (y_2 = 0), \tag{2.7}$$

and the external boundary of this region (because of the absence of its interaction with the external region 1) obeys the same conditions as at the external boundary of the two-dimensional Prandtl boundary layer:

$$u_2 \to 1, \qquad w_2 \to 0, \qquad h_2 \to 1/[(\gamma - 1)\mathrm{M}_\infty^2] \qquad (y_2 \to \infty). \tag{2.8}$$

In addition, the initial conditions satisfied in the cross section $x = x_0$ are

$$u_2 = u_0(y_2), \qquad v_2 = \Delta x A^{1/2}(\gamma - 1)^{1/2}v_0(y_2), \qquad w_2 = 0, \qquad h_2 = h_0(y_2),$$
$$p_2 = -\frac{K}{\gamma - 1}\int\limits_0^{y_2} \rho_0 u_0^2\,dy_2, \qquad \rho_2 = \frac{\rho_0(y_2)}{\gamma - 1}, \qquad \mu_2 = A\mu_0(y_2) \qquad (x_2 = 0), \tag{2.9}$$

and the solution satisfies the condition of periodicity in the transverse direction

$$f(x_2, y_2, z_2) = f(x_2, y_2, z_2 + \lambda), \qquad f \equiv \{u_2, v_2, w_2, p_2, \rho_2, h_2, \mu_2\}, \tag{2.10}$$

where λ is the wavelength of vortices.

The solution of the boundary-value problem (2.6)–(2.10) describes the nonlinear development of long-wave Görtler vortices for $\Delta z \sim (æ\delta)^{1/2}\Delta x \gg \delta$, when they occupy the entire hypersonic boundary layer ($\Delta y \sim \delta$), and the Görtler number is high ($\mathrm{G}_\infty \sim æ/\delta \gg 1$). If the characteristic length of the disturbed vortex region is small as compared to the characteristic length at which the stream functions in the boundary layer change ($\Delta x \ll 1$), the problem becomes local, the evolution of vortices occurs in a plane–parallel flow, Eqs. (2.6) have no dissipative terms, and only no-slip conditions should be taken into account on the concave surface. To satisfy the adhesion conditions near the surface, one can additionally consider the viscous sublayer. If the characteristic length of the vortex region is commensurable with the boundary-layer thickness ($\Delta x \sim 1$), the "growth" of the boundary layer should be taken into account [3].

In (2.6)–(2.10), the variables x_2, y_2, z_2, v_2, w_2, ρ_2, μ_2, and p_2 are related to the quantities $\lambda/(2\pi K^{1/2}A^{1/4}(\gamma-1)^{1/4})$, $A^{1/2}(\gamma-1)^{1/2}$, $\lambda/(2\pi)$, $2\pi K^{1/2}A^{3/4}(\gamma-1)^{3/4}/\lambda$, $K^{1/2}A^{1/4}(\gamma-1)^{1/4}$, $1/(\gamma-1)$, A, and $KA^{1/2}/(\gamma-1)^{1/2}$, respectively; the vertical scale of the disturbed vortex region is the same quantity $A^{1/2}(\gamma-1)^{1/2}$ that is used for the vertical coordinate of the boundary layer n near the concave surface (1.4). In the new variables (the subscript 2 is omitted), the boundary-value problem acquires the form

$$\begin{gathered}
\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0,\\
\rho\Big(u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z}\Big) = \frac{\Delta x}{\mathrm{Re}}\,\frac{\partial}{\partial y}\Big(\mu\,\frac{\partial u}{\partial y}\Big),\\
\frac{\delta}{æ}\,\rho\Big(u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + w\,\frac{\partial v}{\partial z}\Big) + \Delta x^2\lambda_1^2\Big(\rho u^2 + \frac{\partial p}{\partial y}\Big) = 0,\\
\rho\Big(u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + w\,\frac{\partial w}{\partial z}\Big) + \frac{\partial p}{\partial z} = \frac{\Delta x}{\mathrm{Re}}\,\frac{\partial}{\partial y}\Big(\mu\,\frac{\partial w}{\partial y}\Big),\\
\rho\Big(u\,\frac{\partial h}{\partial x} + v\,\frac{\partial h}{\partial y} + w\,\frac{\partial h}{\partial z}\Big) = \frac{\Delta x}{\mathrm{Re}}\Big[\frac{1}{\mathrm{Pr}}\,\frac{\partial}{\partial y}\Big(\mu\,\frac{\partial h}{\partial y}\Big) + \mu\Big(\frac{\partial u}{\partial y}\Big)^2\Big],\\
\rho h = 1, \qquad \mu = h,\\
u = v = w = 0, \qquad h = h_w \quad \text{or} \quad \frac{\partial h}{\partial y} = 0 \qquad (y = 0),\\
u \to 1, \qquad w \to 0, \qquad h \to 1/[(\gamma-1)\mathrm{M}_\infty^2] \qquad (y \to \infty),\\
u = u_0(y), \qquad v = \frac{\Delta x}{\mathrm{Re}}\,v_0(y), \qquad w = 0, \qquad p = -\int_0^y \rho_0 u_0^2\,dy,\\
\rho = \rho_0(y), \qquad \mu = \mu_0(y), \qquad h = h_0(y) \qquad (x = 0),\\
f(x, y, z) = f(x, y, z + 2\pi), \qquad f \equiv \{u, v, w, p, \rho, h, \mu\},\\
\mathrm{Re} = 2\pi K^{1/2}A^{1/4}(\gamma-1)^{1/4}/\lambda, \qquad \lambda_1 = \lambda/(2\pi A^{1/2}(\gamma-1)^{1/2}),
\end{gathered} \tag{2.11}$$

where $\mathrm{Re} \sim 1$ is the local Reynolds number and $\lambda_1 \geqslant 1$ is the ratio of the vortex wavelength to the boundary-layer thickness.

3. We assume that the characteristic streamwise size of the disturbed vortex region is $\Delta x \sim (\delta/æ)^{1/2} \ll 1$. Then it follows from (2.3) that its transverse dimensions are identical in order of magnitude $\Delta y \sim \Delta z \sim \delta$. The ratio of the vortex wavelength to the boundary-layer thickness may be prescribed by the parameter $\lambda_1 \geqslant 1$. In this case, the longitudinal change in the stream functions in the boundary layer is insignificant, the evolution of vortices occurs in a plane–parallel flow, Eqs. (2.11) have no dissipative terms, and only no-slip conditions should be satisfied at the concave surface. Under these conditions, we obtain

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0, \qquad u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z} = 0,$$

$$\rho\Big(u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + w\,\frac{\partial v}{\partial z}\Big) + \lambda_1^2\Big(\rho u^2 + \frac{\partial p}{\partial y}\Big) = 0, \qquad \rho\Big(u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + w\,\frac{\partial w}{\partial z}\Big) + \frac{\partial p}{\partial z} = 0,$$

$$u\,\frac{\partial h}{\partial x} + v\,\frac{\partial h}{\partial y} + w\,\frac{\partial h}{\partial z} = 0, \qquad \rho h = 1, \tag{3.1}$$

$$v = 0 \quad (y = 0), \qquad u \to 1, \quad w \to 0, \quad h \to 1/[(\gamma - 1)\mathrm{M}_\infty^2] \quad (y \to \infty),$$

$$u \to u_0(y), \quad v \to 0, \quad w \to 0, \quad p \to -\int\limits_0^y \rho_0 u_0^2\,dy, \quad \rho \to \rho_0(y), \qquad h \to h_0(y) \quad (x \to -\infty),$$

$$f(x, y, z) = f(x, y, z + 2\pi), \qquad f \equiv \{u, v, w, p, \rho, h\}.$$

For small perturbations of the boundary-layer flow, the boundary-value problem may be linearized with respect to the initial conditions:

$$u = u_0(y) + \alpha U + \ldots, \qquad v = \alpha V + \ldots, \qquad w = \alpha W + \ldots,$$
$$p = -\int\limits_0^y \frac{u_0^2}{h_0}\,dy + \alpha P + \ldots, \quad h = h_0(y) + \alpha H + \ldots, \quad \rho = \frac{1}{h_0(y)} - \alpha\frac{H}{h_0^2(y)} + \ldots, \quad \alpha \ll 1 \tag{3.2}$$

[the last relation in (3.1) for density ρ is taken into account here]. After linearization (3.2), normal-mode representation of the solution [18], and introduction of a new vertical variable

$$F(x, y, z) = F(y)\exp(\beta x)(\sin z, \cos z), \qquad n_0(x_0, y) = \frac{1}{\sqrt{2x_0}}\int\limits_0^y \frac{dy}{h_0} \tag{3.3}$$

the boundary-value problem (3.1) reduces to one equation in ordinary derivatives for the function $V_1 = V/u_0$:

$$V_1'' + 2\Big(\frac{u_0'}{u_0} - \frac{h_0'}{h_0}\Big)V_1' - \frac{h_0^2}{\Lambda^2}\,V_1 = \frac{1}{B^2}\Big(h_0' - \frac{2u_0'h_0}{u_0}\Big)V_1,$$
$$V_1'(0) = V_1(\infty) = 0, \qquad \Lambda = \frac{\lambda_1}{(2x_0)^{1/2}}, \qquad B = \frac{\beta}{(2x_0)^{1/4}}, \qquad (\,\cdot\,)' = \frac{d}{dn_0}\,. \tag{3.4}$$

The solution of the boundary-value problem (3.4) allows determination of its eigenfunctions $V(n_0)$ (vertical component of the vortex velocity) and eigenvalues B (increment of the vortex amplitude).

The numerical solution of (3.4) is obtained by the method of inverse iterations with shifting [19] for the first three vortex modes and different functions $u_0(n_0)$ and $h_0(n_0)$ in the boundary layer; the specific heat capacity is $\gamma = 1.4$.

Figure 1a and b shows the increments B_1 and B_2 for the first two modes as functions of the relative wavelength of the vortices $\Lambda \geqslant 1$ for Pr = 1 and M_∞ = 5, 15.8, and 50 (curves 2–4, respectively). The surface temperature is 10 times the free-stream temperature ($T_w = 10T_\infty$), which corresponds to the ultimate strength of the surface material. Curve 1 is calculated for the boundary layer in a fluid. A principal difference in the behavior of the dependences for the first mode and all subsequent modes should be noted. First, the increment B_1 increases with increasing M_∞ for all values of Λ, whereas the increments of higher modes B_m ($m = 2, 3, \ldots$), vice versa, decrease with increasing Mach number (except for a range of small values of M_∞ and Λ, which decreases with increasing mode number m). Second, for all values of M_∞, the increment B_1 increases monotonically with increasing Λ, and the increments B_m ($m = 2, 3, \ldots$) are almost independent of Λ (except for the same range of the values of M_∞ and Λ). A similar behavior of vortices was obtained from an analytical solution [5, 8] in studying centrifugal instability of the boundary layer of a fluid with intense suction [20], where the layer thickness is constant:

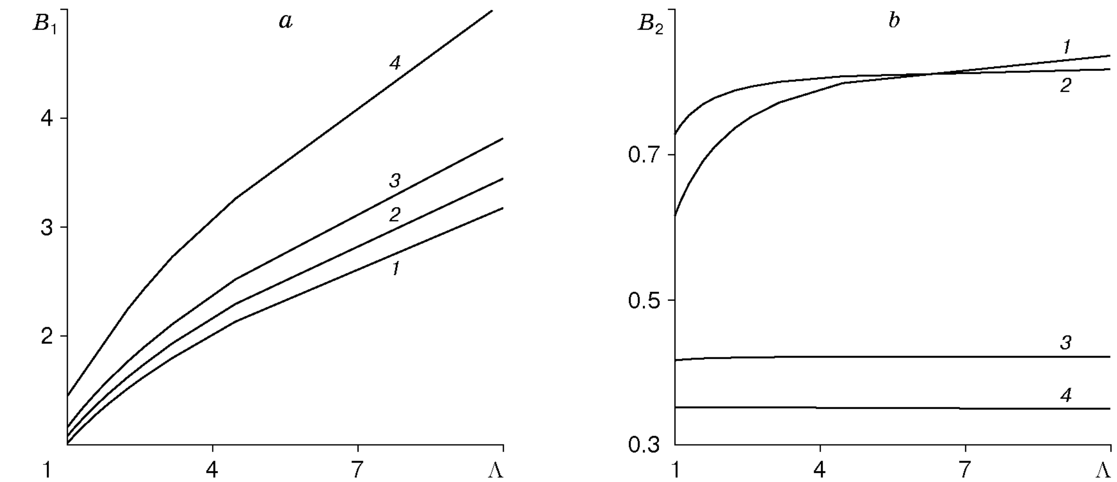

Fig. 1

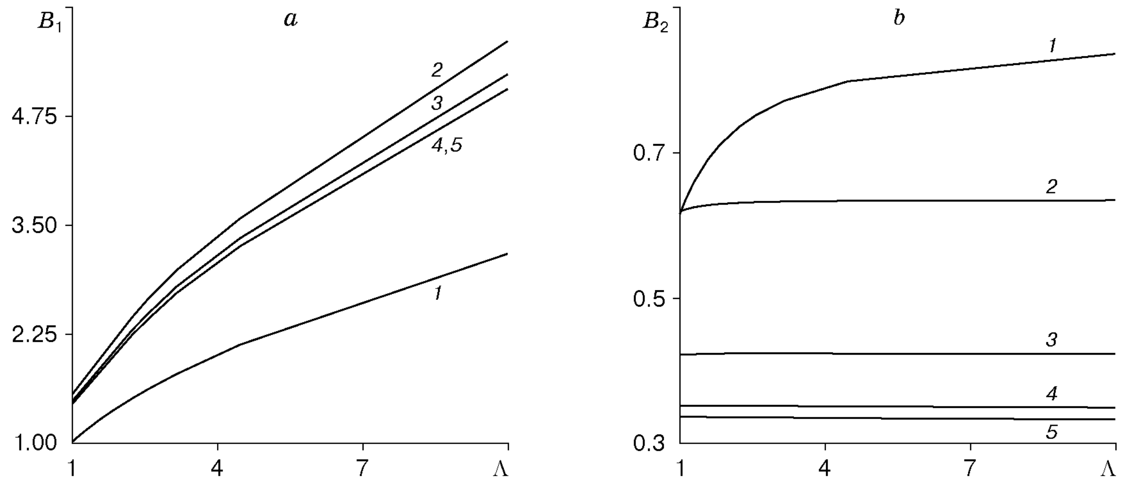

Fig. 2

$B_1 = \Lambda^{1/2}$ and $B_m \approx \sqrt{2/(m^2 - 1)}$ ($m = 2, 3, \ldots$ and $\Lambda \gg 1$). Such a character of variation of B_1 means that the first long-wave mode is separated from the subsequent modes as the vortex wavelength increases, and its linear growth should occur at smaller characteristic distances.

An increase in M_∞ leads to heating of the boundary layer, increase in its thickness ($\delta \sim \mathrm{M}_\infty^2/\mathrm{Re}_\infty^{1/2} \sim \mathrm{M}_\infty^{3/2}\delta_\mathrm{f}$) and characteristic length of the disturbed vortex region [$\Delta x \sim (\delta/æ)^{1/2} \sim \mathrm{M}_\infty^{3/4}\Delta x_\mathrm{f}$] as compared to the boundary layer of a fluid (the subscript "f" refers to fluid parameters). This induces a decrease in the vortex growth rate reduced to the characteristic length (of the order of unity)

$$Be \sim B/\Delta x \sim B/(\mathrm{M}_\infty^{3/4}\Delta x_\mathrm{f}),$$

since an increase in the numerator is insignificant, and an increase in the denominator plays the determining role [14]. The same fact is responsible for the stabilizing effect of the increase in the Mach number M_∞ on long-wave Görtler vortices.

Figure 2a and b shows the effect of the dimensionless surface enthalpy $h_0(0)$ on the increments of two first modes B_1 and B_2 for $\mathrm{M}_\infty = 50$ and $\mathrm{Pr} = 1$ [curve 1 refers to the fluid and curve 2 refers to a heat-insulated surface with $h_0'(0) = 0$; curves 3–5 are calculated foe $h_0(0) = 0.1$, 0.01, and 0, respectively]. Surface heating leads to an insignificant increase in the first-mode increment B_1; the increments of higher modes B_m ($m = 2, 3, \ldots$) increase approximately twofold from the values for an absolutely cold surface with $h_0(0) = 0$ to the values for a heat-insulated surface with $h_0'(0) = 0$. It is known, however, that surface heating increases the boundary-layer thickness [20] and, hence, the characteristic dimensions of the vortices. Therefore, it is not possible to evaluate unambiguously the influence of surface heating on the vortex growth rate Be. (A weak stabilizing effect of surface heating on long-wave vortices was noted in [9, 11, 12].)

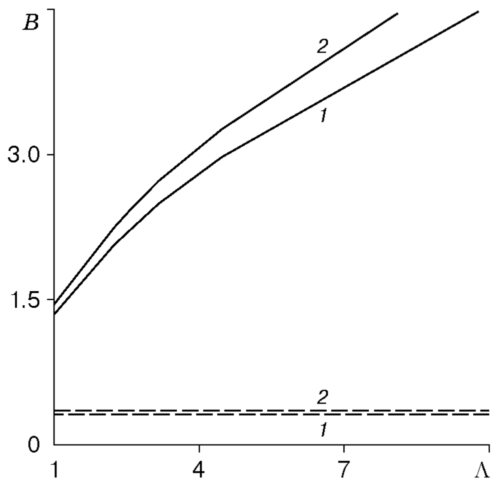

Fig. 3

Figure 3 shows the effect of the Prandtl number Pr on the increments B for two higher modes (the solid and dashed curves refer to B_1 and B_2, respectively) for $\mathrm{M}_\infty = 50$ and $T_w = 10\,T_\infty$. An increase in Pr from 0.5 to 1 (curves 1 and 2) leads to an increase in the increment for the first mode and exerts practically no effect on the increments of higher modes (the curves obtained for the third mode are almost indiscernible and are not plotted in Fig. 3).

According to estimates (2.3), the characteristic length of the vortex region is proportional to the relative wavelength of the vortices; therefore, the vortex growth rate $Be \sim B/\Delta x \sim B/\Lambda$ decreases with increasing Λ (the governing quantity in this fraction is also the denominator).

The profiles of the eigenfunctions $V(n_0)$ for $\Lambda = 10$ for the first, second, and third modes are presented in Fig. 4a, b, and c, respectively (curves 1 are calculated for a fluid and curves 2–4 are calculated for $\mathrm{M}_\infty = 5$, 15.8, and 50, respectively, and $\mathrm{Pr} = 1$ and $T_w = 10\,T_\infty$). It follows from Eq. (3.4) that $V \sim \exp(-h_0 n_0/\Lambda)$ as $n_0 \to \infty$. Therefore, the decay of eigenfunctions significantly decreases with increasing M_∞ (since $h_0 \sim 1/\mathrm{M}_\infty^2$ as $n_0 \to \infty$) and relative wavelength of the vortices Λ. Nevertheless, an increase in M_∞ leads to a corresponding extension of the vertical coordinate n_0 [see (3.3)] and does not increase the values of the physical variable y at which vortex decay occurs. However, as the wavelength increases, the vortices at the linear stage of development go further outside the external edge of the boundary layer and decay at distances $\Delta y \sim \Delta z \sim (æ\delta)^{1/2}\Delta x \gg \delta$. This fact was noted in [3]; it is associated with the use of the normal-mode representation of solution (3.3).

4. We consider the evolution of long-wave vortices that introduce only small perturbations (for instance, $\Delta u \ll u \sim 1$) into the main part of the boundary layer (region 2) and may induce nonlinear perturbations (for example, $\Delta u \sim u \ll 1$) in its near-wall part (region 3). We assume that the friction coefficient

$$C_\tau = \frac{\delta}{\mathrm{M}_\infty^2}\, c = \frac{\mu}{\mathrm{Re}_\infty}\Big(\frac{\partial u}{\partial y}\Big)_w$$

and the heat-transfer coefficient

$$C_q = \frac{\delta}{\mathrm{M}_\infty^2}\,\frac{b}{\mathrm{Pr}} = \frac{\mu}{\mathrm{Re}_\infty \mathrm{Pr}}\Big(\frac{\partial h}{\partial y}\Big)_w$$

(b and c are constants) retain their orders of magnitude in region 3 for $\Delta y/\delta \ll 1$. Then, from Eqs. (1.2) and (1.3), we can find the distributions of the longitudinal component of velocity u and enthalpy h in this part of the undisturbed boundary layer:

$$u = \frac{c}{b}\Big(\frac{2b}{A}\,\frac{\Delta y}{\delta} + h_w^2\Big)^{1/2} - \frac{c}{b}\,h_w, \qquad h = \Big(\frac{2b}{A}\,\frac{\Delta y}{\delta} + h_w^2\Big)^{1/2},$$

$$u \approx \frac{c}{Ah_w}\,\frac{\Delta y}{\delta}, \qquad h \approx h_w + \frac{b}{Ah_w}\,\frac{\Delta y}{\delta} \quad \text{for} \quad \Big(\frac{\Delta y}{\delta}\Big)^{1/2} \ll h_w \leqslant 1. \tag{4.1}$$

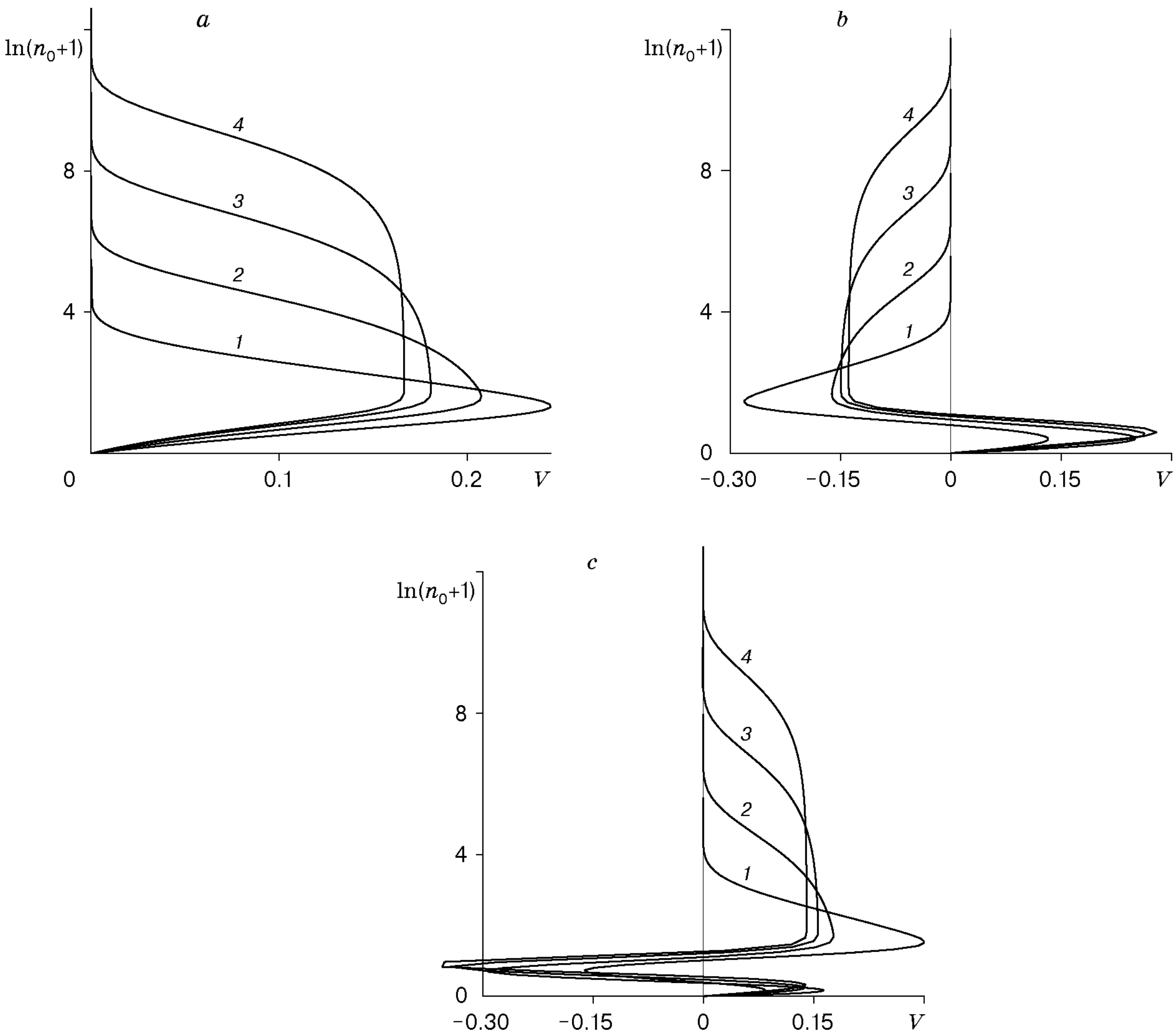

Fig. 4

Assuming that the flow in region 3 is viscous, spatial, and nonlinear, by comparing the main terms of the Navier–Stokes equations [$\mathrm{Re}_\infty \rho u\, \partial u/\partial x \sim \partial(\mu\, \partial u/\partial y)/\partial y$, $\partial(\rho u)/\partial x \sim \partial(\rho v)/\partial y \sim \partial(\rho w)/\partial z$, and $\Delta p \sim \rho w^2$] and using the approximate relations (4.1), we can obtain estimates of the thickness of region 3, vertical and transverse components of velocity, and pressure perturbation:

$$\Delta y \sim h_w \delta \Delta x^{1/3}, \qquad v \sim \frac{h_w \delta}{\Delta x^{1/3}}, \qquad w \sim \frac{\Delta z}{\Delta x^{2/3}}, \qquad \Delta p \sim \frac{\Delta z^2}{\mathrm{M}_\infty^2 h_w \Delta x^{4/3}}. \tag{4.2}$$

Estimates (4.2) allow introduction of new variables $x = x_0 + \Delta x x_3$, $y = h_w \delta \Delta x^{1/3} y_3$, and $z = \Delta z z_3$ and asymptotic expansions of the stream functions for the near-wall part of the disturbed vortex region:

$$u = \Delta x^{1/3} u_3 + \ldots, \qquad v = \frac{h_w \delta}{\Delta x^{1/3}}\, v_3 + \ldots, \qquad w = \frac{\Delta z}{\Delta x^{2/3}}\, w_3 + \ldots,$$

$$p = \frac{1}{\gamma \mathrm{M}_\infty^2} + \ldots + \frac{\Delta z^2}{\mathrm{M}_\infty^2 h_w \Delta x^{4/3}}\, p_3 + \ldots, \qquad \rho = \frac{\rho_w}{\mathrm{M}_\infty^2 h_w} + \ldots, \tag{4.3}$$

$$h = h_w + \Delta x^{1/3} h_3 + \ldots, \qquad \mu = \mathrm{M}_\infty^2 h_w \mu_w + \ldots \,.$$

Substituting expansions (4.3) into the Navier–Stokes equations and into Eqs. (1.2) and (1.3) and performing the limiting transition (2.5) for $\Delta x^{1/3} \ll h_w \leqslant 1$, we find that the flow in region 3 in the first approximation is described by a system of equations for an incompressible fluid:

$$\frac{\partial u_3}{\partial x_3} + \frac{\partial v_3}{\partial y_3} + \frac{\partial w_3}{\partial z_3} = 0, \qquad \frac{\partial p_3}{\partial y_3} = 0,$$

$$\rho_w\Big(u_3 \frac{\partial u_3}{\partial x_3} + v_3 \frac{\partial u_3}{\partial y_3} + w_3 \frac{\partial u_3}{\partial z_3}\Big) = \mu_w \frac{\partial^2 u_3}{\partial y_3^2},$$

$$\rho_w\Big(u_3 \frac{\partial w_3}{\partial x_3} + v_3 \frac{\partial w_3}{\partial y_3} + w_3 \frac{\partial w_3}{\partial z_3}\Big) + \frac{\partial p_3}{\partial z_3} = \mu_w \frac{\partial^2 w_3}{\partial y_3^2}, \tag{4.4}$$

$$\rho_w\Big(u_3 \frac{\partial h_3}{\partial x_3} + v_3 \frac{\partial h_3}{\partial y_3} + w_3 \frac{\partial h_3}{\partial z_3}\Big) = \frac{\mu_w}{\mathrm{Pr}} \frac{\partial^2 h_3}{\partial y_3^2},$$

$$(\gamma - 1)\rho_w = 1, \qquad \mu_w = A.$$

Usual no-slip and adhesion conditions are satisfied at the concave surface:

$$u_3 = v_3 = w_3 = h_3 = 0 \qquad (y_3 = 0); \tag{4.5}$$

the initial conditions for this region are obtained by matching with the solution for the near-wall part of the undisturbed boundary layer (4.1)

$$u_3 \to (c/A)y_3, \qquad h_3 \to (b/A)y_3, \qquad v_3, w_3, p_3 \to 0 \qquad (x_3 \to -\infty). \tag{4.6}$$

In region 2 with a characteristic thickness $\Delta y \sim \delta$, where the orders of magnitude of the longitudinal component of velocity u, enthalpy h, density ρ, and viscosity μ are determined from (1.1), the pressure perturbation Δp is generated by centrifugal effects and has the same order of magnitude as in region 3:

$$\Delta p \sim k\rho u \Delta u \Delta y \sim æ\delta \Delta u/\mathrm{M}_\infty^2 \sim \Delta z^2/(\mathrm{M}_\infty^2 h_w \Delta x^{4/3}).$$

Then, by comparing the orders of magnitude of the main terms of the Navier–Stokes equations, we obtain estimates for perturbations of the velocity components:

$$\Delta u \sim \Delta z^2/(h_w \Delta x^{4/3} æ\delta), \qquad v \sim \Delta z^2/(h_w \Delta x^{7/3} æ), \qquad w \sim \Delta z/(h_w \Delta x^{1/3}). \tag{4.7}$$

Estimates (4.7) allow us to introduce new variables $x = x_0 + \Delta x x_3$, $y = \delta y_2$, and $z = \Delta z z_3$ and asymptotic expansions of the stream functions for the main part of the boundary layer:

$$u = u_0(y_2) + \frac{\Delta z^2}{h_w \Delta x^{4/3} æ\delta} u_2 + \ldots, \qquad v = \frac{\Delta z^2}{h_w \Delta x^{7/3} æ} v_2 + \ldots, \qquad w = \frac{\Delta z}{h_w \Delta x^{1/3}} w_2 + \ldots,$$

$$p = \frac{1}{\gamma \mathrm{M}_\infty^2} + \ldots - \frac{æ\delta}{\mathrm{M}_\infty^2} K \int\limits_0^{y_2} \rho_0 u_0^2 \, dy_2 + \frac{\Delta z^2}{\mathrm{M}_\infty^2 h_w \Delta x^{4/3}} p_2 + \ldots, \tag{4.8}$$

$$\rho = \frac{\rho_0(y_2)}{\mathrm{M}_\infty^2} + \frac{\Delta z^2}{\mathrm{M}_\infty^2 h_w \Delta x^{4/3} æ\delta} \rho_2 + \ldots, \qquad h = h_0(y_2) + \frac{\Delta z^2}{h_w \Delta x^{4/3} æ\delta} h_2 + \ldots$$

(the subscript 0 refers to flow parameters in the undisturbed boundary layer).

Substituting expansions (4.8) into the Navier–Stokes equations and into Eqs. (1.2) and (1.3) and performing the limiting transition (2.5) for $\Delta x^{1/3} \ll h_w \leqslant 1$ and $\delta^{1/2}/æ^{1/2} \ll \Delta x \ll \Delta z/(æ^{1/2}\delta^{1/2})$, we find that the flow in region 2 in the first approximation is described by the system of linear equations

$$\rho_0 \frac{\partial u_2}{\partial x_3} + u_0 \frac{\partial \rho_2}{\partial x_3} + \rho_0 \frac{\partial v_2}{\partial y_2} + v_2 \frac{d\rho_0}{dy_2} = 0,$$

$$u_0 \frac{\partial u_2}{\partial x_3} + v_2 \frac{du_0}{dy_2} = 0, \qquad 2K\rho_0 u_0 u_2 + K\rho_2 u_0^2 + \frac{\partial p_2}{\partial y_2} = 0,$$

$$\rho_0 u_0 \frac{\partial w_2}{\partial x_3} + \frac{\partial p_2}{\partial z_3} = 0, \qquad u_0 \frac{\partial h_2}{\partial x_3} + v_2 \frac{dh_0}{dy_2} = 0,$$

$$(\gamma - 1)\rho_0 h_0 = 1, \qquad \rho_0 h_2 + \rho_2 h_0 = 0,$$

which allows partial integration

$$u_2 = D\,\frac{du_0}{dy_2}, \qquad v_2 = -u_0\,\frac{\partial D}{\partial x_3}, \qquad \rho_2 = D\,\frac{d\rho_0}{dy_2}, \qquad h_2 = D\frac{dh_0}{dy_2},$$
$$p_2 = p_2\Big|_{y_2\to\infty} + KD(\mathrm{M}_\infty^2 - \rho_0 u_0^2), \qquad D = D(x_3, z_3). \tag{4.9}$$

It follows from the last relation in (4.9) that the pressure perturbation p_2 increases toward the external edge of the boundary layer due to centrifugal effects in region 2 ($p_2|_{y_2\to\infty} \sim \mathrm{M}_\infty^2$). Therefore, in the disturbed region 1 of the uniform external flow with characteristic dimensions $\delta \ll \Delta y \sim \Delta z \ll \Delta x \ll 1$, where the longitudinal component of velocity is $u \sim 1$, the gas density is $\rho \sim 1$, and the gas enthalpy is $h \sim 1/((\gamma-1)\mathrm{M}_\infty^2)$, the pressure perturbation has the following order of magnitude:

$$\Delta p \sim \Delta z^2/(h_w \Delta x^{4/3}).$$

A comparison of the orders of magnitude of the main terms of the Navier–Stokes equations allows us to obtain estimates for perturbations of the stream functions and introduce in region 1 new variables $x = x_0 + \Delta x x_3$, $y = \Delta z y_1$, and $z = \Delta z z_3$ and asymptotic expansions

$$u = 1 + \frac{\Delta z^2}{h_w \Delta x^{4/3}}\,u_1 + \ldots, \qquad v = \frac{\Delta z}{h_w \Delta x^{1/3}}\,v_1 + \ldots, \qquad w = \frac{\Delta z}{h_w \Delta x^{1/3}}\,w_1 + \ldots,$$
$$p = \frac{1}{\gamma \mathrm{M}_\infty^2} + \ldots - æ\Delta z K y_1 + \frac{\Delta z^2}{h_w \Delta x^{4/3}}\,p_1 + \ldots, \tag{4.10}$$
$$\rho = 1 + \frac{\Delta z^2}{h_w \Delta x^{4/3}}\,\rho_1 + \ldots, \qquad h = \frac{1}{(\gamma-1)\mathrm{M}_\infty^2} + \frac{\Delta z^2}{h_w \Delta x^{4/3}}\,h_1 + \ldots\,.$$

Substituting expansions (4.10) into the Navier–Stokes equations and into Eqs. (1.2) and (1.3) and performing the limiting transitions (2.5) for $\Delta x^{1/3} \ll h_w \leqslant 1$ and $\delta^{1/2}/æ^{1/2} \ll \Delta x \ll \Delta z/(æ^{1/2}\delta^{1/2})$, we find that the following equations are valid in the first approximation in region 1:

$$\frac{\Delta z^2}{\Delta x^2}\Big(\frac{\partial u_1}{\partial x_3} + \frac{\partial \rho_1}{\partial x_3}\Big) + \frac{\partial v_1}{\partial y_1} + \frac{\partial w_1}{\partial z_3} = 0,$$
$$\frac{\partial u_1}{\partial x_3} + \frac{\partial p_1}{\partial x_3} = 0, \qquad \frac{\partial v_1}{\partial x_3} + \frac{\partial p_1}{\partial y_1} = 0, \qquad \frac{\partial w_1}{\partial x_3} + \frac{\partial p_1}{\partial z_3} = 0, \qquad \frac{\partial h_1}{\partial x_3} - \frac{\partial p_1}{\partial x_3} = 0, \tag{4.11}$$
$$\mathrm{M}_\infty^2(\gamma-1)h_1 + \rho_1 = \gamma \mathrm{M}_\infty^2 p_1.$$

These equations may be transformed to the wave equation for the pressure perturbation p_1:

$$\frac{\mathrm{M}_\infty^2}{\mathrm{M}_c^2}\,\frac{\partial^2 p_1}{\partial x_3^2} = \frac{\partial^2 p_1}{\partial y_1^2} + \frac{\partial^2 p_1}{\partial z_3^2}, \qquad \mathrm{M}_c \sim \frac{\Delta x}{\Delta z} \gg 1. \tag{4.12}$$

The ratio of the characteristic longitudinal and transverse dimensions of the disturbed vortex region determines the critical Mach number M_c (for $\mathrm{M}_\infty \sim \mathrm{M}_c$, the characteristic dimensions of the disturbed vortex region in the order of magnitude are equal to the Mach cone dimensions).

Matching of the asymptotic expansions (4.3) and (4.8) in regions 3 and 2, Eqs. (4.1) and (4.9) taken into account, allows us to obtain the missing external boundary conditions for region 3 and the condition for their nontrivial interaction:

$$u_3 \to (c/A)(y_3 + D), \qquad w_3 \to 0, \qquad h_3 \to (b/A)(y_3 + D) \qquad (y_3 \to \infty),$$
$$p_3 = p_2|_{y_2=0} = p_2|_{y_2\to\infty} + KD\mathrm{M}_\infty^2, \qquad \Delta z \sim h_w æ^{1/2}\delta^{1/2}\Delta x^{5/6}. \tag{4.13}$$

For nontrivial interaction of regions 2 and 1, the order of magnitude of the vertical component of velocity v should be retained constant in these regions [16]. This is possible under the condition

$$\Delta z \sim æ\Delta x^2. \tag{4.14}$$

Matching the asymptotic expansions (4.8) and (4.10), Eqs. (4.9), (4.11), (4.13), and (4.14) taken into account, we obtain

$$v_1|_{y_1=0} = v_2|_{y_2\to\infty} = -\frac{\partial D}{\partial x_3}, \qquad p_2|_{y_2\to\infty} = \mathrm{M}_\infty^2 p_1|_{y_1=0},$$
$$p_3 = \mathrm{M}_\infty^2 (p_1|_{y_1=0} + KD), \qquad \frac{\partial p_1}{\partial y_1} = \frac{\partial^2 D}{\partial x_3^2} \qquad (y_1 = 0). \tag{4.15}$$

If conditions (4.13) and (4.14) are simultaneously satisfied, a full three-dimensional flow structure is formed [5–8, 21] in the disturbed vortex region with characteristic dimensions

$$\Delta x \sim h_w^{6/7} (\delta/æ)^{3/7}, \qquad \Delta z \sim h_w^{12/7} æ^{1/7} \delta^{6/7}. \tag{4.16}$$

In this case, the flow in the near-wall region 3 is described by Eqs. (4.4) and the boundary and initial conditions (4.5), (4.6), and (4.13). The pressure perturbation p_3 is determined by Eq. (4.15); it is composed of pressure perturbations due to the displacing action of the boundary layer ($p_1|_{y_1=0}$) and centrifugal effects (KD). These components are found from the joint solution of the problem for region 3 and the wave equation (4.12) for region 1, condition (4.15) at the internal boundary and conditions of restriction of disturbances at the external boundaries being satisfied. Naturally, the solutions in regions 3 and 1 should satisfy the condition of periodicity in the transverse direction [see (2.10)].

The variables y_1, p_1, x_3, y_3, z_3, u_3, v_3, w_3, p_3, h_3, and D are normalized to the quantities $\lambda/(2\pi)$, $\lambda^3 A^6 (\gamma-1)^3/(8\pi^3 K \mathrm{M}_\infty^2 c^2 l^{10})$, $cl^3/(A^2(\gamma-1))$, l, $\lambda/(2\pi)$, cl/A, $A(\gamma-1)/l$, $\lambda A(\gamma-1)/(2\pi l^2)$, $\lambda^2 A^2(\gamma-1)/(4\pi^2 l^4)$, bl/A, and $\lambda^2 A^2 (\gamma-1)/(4\pi^2 K \mathrm{M}_\infty^2 l^4)$, respectively. Then, the boundary-value problem (2.10), (4.4)–(4.6), (4.13), (4.15) for region 3 acquires the following form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \qquad u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial p}{\partial y} = 0,$$
$$u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + w\,\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} = \frac{\partial^2 w}{\partial y^2}, \qquad u\,\frac{\partial h}{\partial x} + v\,\frac{\partial h}{\partial y} + w\,\frac{\partial h}{\partial z} = \frac{1}{\mathrm{Pr}}\,\frac{\partial^2 h}{\partial y^2},$$
$$u = v = w = h = 0 \qquad (y = 0),$$
$$u, h \to y + \gamma_1 D, \quad w \to 0 \quad (y \to \infty), \qquad u, h \to y, \quad v, w, p, D \to 0 \quad (x \to -\infty), \tag{4.17}$$
$$p = \gamma_2 p_1|_{y_1=0} + D,$$
$$u(x,y,z) = u(x,y,z+2\pi), \quad v(x,y,z) = v(x,y,z+2\pi), \quad w(x,y,z) = w(x,y,z+2\pi),$$
$$h(x,y,z) = h(x,y,z+2\pi), \quad p(x,z) = p(x,z+2\pi), \quad D(x,z) = D(x,z+2\pi),$$
$$\gamma_1 = \lambda^2 A^2(\gamma-1)/(4\pi^2 l^5 K \mathrm{M}_\infty^2), \qquad \gamma_2 = \lambda A^4 (\gamma-1)^2/(2\pi l^6 K c^2)$$

(notation is the same, the subscript 3 is omitted, and the thickness of layer 3 is normalized to an arbitrary quantity l). The boundary-value problem for region 1 is

$$\gamma_3\,\frac{\partial^2 p_1}{\partial x^2} = \frac{\partial^2 p_1}{\partial y_1^2} + \frac{\partial^2 p_1}{\partial z^2}, \qquad \frac{\partial p_1}{\partial y_1} = \frac{\partial^2 D}{\partial x^2} \qquad (y_1 = 0),$$
$$p_1(x, y_1, z) = p_1(x, y_1, z+2\pi), \qquad \gamma_3 = \lambda^2 A^4 (\gamma-1)^2 \mathrm{M}_\infty^2/(4\pi^2 l^6 c^2 \mathrm{M}_c^2). \tag{4.18}$$

The boundary-value problems (4.17) and (4.18) contain three parameters of similarity: γ_1, γ_2, and γ_3. The first parameter determines the degree of interaction of regions 3 and 2, the second parameter determines the same for regions 2 and 1, and the third parameter is responsible for the regime of interaction of viscid and inviscid flows. The latter circumstance distinguishes the problem considered here of a three-layered perturbation of the flow, which arises in the development of long-wave Görtler vortices in a hypersonic boundary layer near a moderately cooled concave surface, from a similar problem for an incompressible fluid [5–8]. Therefore, the equation for perturbation of the enthalpy h in (4.17) may be solved independently.

Subsequent linearization of the solution of the boundary-value problem (4.17) with respect to the initial conditions and the use of the normal-mode representation of solutions (3.3) in regions 3 and 1 allow us to reduce problems (4.17) and (4.18) to a system of ordinary differential equations

$$\beta U + V' + W = 0, \qquad \beta y U + V = U'',$$

$$\beta y W - D[1 - \gamma_2\beta^2/(1 + \gamma_3\beta^2)^{1/2}] = W'', \tag{4.19}$$

$$U = V = W = 0 \quad (y = 0), \qquad U \to \gamma_1 D, \quad W \to 0 \quad (y \to \infty),$$

where D is a constant coefficient in representation of the form (3.3); $(\,\cdot\,)' = d/dy$.

System (4.19) may be reduced to one equation for the function $\beta U' + W'$, whose solution is expressed via the Airy function $\mathrm{Ai}(y/\beta^{1/3})$ [22]. The following dispersion relation is valid:

$$\gamma_2\beta^2/(1 + \gamma_3\beta^2)^{1/2} - 3\gamma_1 \mathrm{Ai}'(0)\beta^{5/3} = 1. \tag{4.20}$$

Relation (4.20) differs from the corresponding relation for a fluid only by the presence of the parameter γ_3 (see, for example, [5, 8]). Nevertheless, it follows from (4.12) and (4.16) that the critical Mach number is $\mathrm{M}_c \sim h_w^{-6/7}$; therefore, we have $\gamma_3 \sim h_w^{12/7}$. Surface cooling and a decrease in h_w lead to a decrease in the parameter γ_3, relation (4.20) acquires a form corresponding to an incompressible fluid, the increment β is independent of h_w, and the vortex growth rate normalized to the characteristic length (of the order of unity)

$$Be \sim \beta/\Delta x \sim (\beta/h_w^{6/7})(æ/\delta)^{3/7}$$

increases due to both surface cooling and decreasing boundary-layer thickness δ. Thus, it is shown analytically that surface cooling leads to an increase in the growth rate of long-wave Görtler vortices.

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